1. Show that $\|x+y\| = \|x\| + \|y\|$ if and only if one of the vectors x, y is a non negative scalar multiple of the other, where x, y are in an inner product space.

Ans: Let $\|x+y\| = \|x\| + \|y\|$

$\therefore \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$

or, $\|x\|^2 + \|y\|^2 + \langle x,y\rangle + \langle y,x\rangle = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$

So $2\,\mathrm{Re}\langle x,y\rangle = 2\|x\|\|y\|$

or, $\mathrm{Re}\langle x,y\rangle = \|x\|\|y\|$.

Let $z = y - \frac{\|y\|}{\|x\|}x$. Then $\langle z,z\rangle = \left(y - \frac{\|y\|}{\|x\|}x, y - \frac{\|y\|}{\|x\|}x\right)$

$= \|y\|^2 - \frac{\|y\|}{\|x\|}\langle y,x\rangle - \frac{\|y\|}{\|x\|}\langle x,y\rangle + \|y\|^2$

$= 2\|y\|^2 - \frac{\|y\|}{\|x\|}(2\,\mathrm{Re}\langle x,y\rangle) = 2\|y\|^2 - 2\|y\|^2 = 0$

So $z = 0 \Rightarrow y = \frac{\|y\|}{\|x\|}x = cx$, $c = \frac{\|y\|}{\|x\|}$ is a non negative real number. If $x = 0$ then $x = 0y$.

Conversely, let $y = cx$, c is, a non negative real number.

Then $\|x+y\| = \|x+cx\| = \|(1+c)x\| = (1+c)\|x\|$

and $\|x\| + \|y\| = \|x\|(1+|c|) = \|x\|(1+c)$

$\therefore \|x+y\| = \|x\| + \|y\|$

2. Using Cauchy-Schwarz inequality, prove that cosine of an angle is of absolute value at most 1

Ans: Let F = Field of real numbers and $V = F^{(3)}$

Consider standard inner product on V,

Let, $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2) \in V$

Let $0 = (0,0,0)$. Let θ be an angle between

OU and OV. Then $\cos\theta = \dfrac{\langle u, v\rangle}{\|u\|\,\|v\|}$

$\therefore\ |\cos\theta| = \dfrac{|\langle u, v\rangle|}{\|u\|\,\|v\|} \leq \dfrac{\|u\|\,\|v\|}{\|u\|\,\|v\|} = 1.$

3. Let V be a non-zero inner product space of dimension n. Then V has an orthonormal basis.

Ans. It is enough to construct an orthogonal basis of V.
Let $S \subseteq V$ be an orthogonal set. Then $T = \left\{\dfrac{x}{\|x\|} : x \in S\right\}$ is an orthonormal set. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V. Let $w_1 = v_1$. Define $w_2 = v_2 - \dfrac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$

$= v_2 - \dfrac{\langle v_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1.$

Then $\langle w_2, w_1\rangle = \langle w_2, v_1\rangle = \langle v_2, v_1\rangle - \dfrac{\langle v_2, v_1\rangle}{\langle v_1, v_1\rangle}\langle v_1, v_1\rangle$
$= 0$

Also $v_2 = \alpha_1 v_1 + w_2 = \alpha_1 w_1 + w_2$; where $\alpha_1 = \dfrac{\langle v_2, v_1\rangle}{\langle v_1, v_1\rangle}$

(Note v_1 is linearly independent $\Rightarrow v_1 \neq \theta \Rightarrow \langle v_1, v_1\rangle \neq 0$). Define $w_3 = v_3 - \dfrac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \dfrac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1$

Then $\langle w_3, w_2\rangle = 0 = \langle w_3, w_1\rangle$

Also $v_3 = \alpha_1 w_1 + \alpha_2 w_2 + w_3$, where $\alpha_1, \alpha_2 \in F$.

In this maner, we can construct an orthogonal set $\{w_1, w_2, \ldots, w_n\}$ where each $v_i = \alpha_1 w_1 + \alpha_2 w_2 + \cdots + w_i$, $\alpha_i \in F$.

$\therefore \left\{\dfrac{w_1}{\|w_1\|}, \ldots, \dfrac{w_n}{\|w_n\|}\right\}$ is an orthonormal set which linearly independent

4. Obtain an orthonormal basis, w.r.to the standard inner product for the subspace of R^3 generated by (1,0,3) and (2,1,1).

Ans: Let $v_1 = (1,0,3)$, $v_2 = (2,1,1)$

Then $w_1 = v_1$, $w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$

Now $\langle v_2, w_1\rangle = \langle v_2, v_1\rangle = 2+0+3 = 5$

$\langle w_1, w_1\rangle = \langle v_1, v_1\rangle = 1+0+9 = 10$.

$\therefore \|w_1\| = \sqrt{10}$, So, $w_2 = (2,1,1) - \frac{5}{10}(1,0,3)$

$= \left(\frac{3}{2}, 1, -\frac{1}{2}\right)$

$\therefore \|w_2\| = \sqrt{\frac{9}{4}+1+\frac{1}{4}} = \sqrt{7/2}$.

$\therefore$ required orthonormal basis is

$$\left\{\frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}\right\} = \left\{\frac{1}{\sqrt{10}}(1,0,3), \sqrt{\frac{2}{7}}\left(\frac{3}{2}, 1, -\frac{1}{2}\right)\right\}$$

5. If $\{w_1, w_2, \ldots, w_m\}$ is an orthonormal set in V,

then $\sum_{i=1}^{m} |\langle w_i, v\rangle|^2 \leq \|v\|^2$ for all $v \in V$

Ans: Let $x = v - \sum_{i=1}^{m}\langle v, w_i\rangle w_i$

$\therefore \langle x, w_j\rangle = \langle v, w_j\rangle - \langle v, w_j\rangle = 0$ for all $j = 1(1)m$.

Let $w = \sum_{i=1}^{m}\langle v, w_i\rangle w_i = \sum_{i=1}^{m}\alpha_i w_i$, $\alpha_i = \langle v, w_i\rangle$

$\therefore v = x + w$.

Also $\langle w, x\rangle = \langle \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_m w_m, x\rangle$

$= \alpha_1\langle w_1, x\rangle + \alpha_2\langle w_2, x\rangle + \cdots + \alpha_m\langle w_m, x\rangle = 0$

Now, $\|v\|^2 = \langle v, v\rangle = \langle w+x, w+x\rangle$

$= \|w\|^2 + \|x\|^2 \geq \|w\|^2$

But $\|w\|^2 = \langle w, w\rangle = \langle \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_m w_m, \alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_m w_m\rangle$

$= \alpha_1\bar{\alpha}_1 \langle w_1, w_1\rangle + \alpha_2\bar{\alpha}_2 \langle w_2, w_2\rangle + \cdots + \alpha_m\bar{\alpha}_m \langle w_m, w_m\rangle$

$= |\alpha_1|^2 + \cdots + |\alpha_m|^2$.

as $\{w_1, w_2, \ldots, w_m\}$ is an orthonormal set.

$= \sum_{i=1}^{m} |\alpha_i|^2 = \sum_{i=1}^{m} |\langle v_i, w_i\rangle|^2 = \sum_{i=1}^{m} |\overline{\langle w_i, v\rangle}|^2 = \sum_{i=1}^{m} |\langle w_i, v\rangle|^2$

$\therefore \sum_{i=1}^{m} |\langle w_i, v\rangle|^2 \le \|v\|^2$ for all $v \in V$.

6. If V is a finite dimensional inner product space and W is a subspace of V, then $V = W \oplus W^\perp$.

Ans: Since V is an inner product space, so W has an orthonormal basis $\{w_1, w_2, \ldots, w_m\}$.

Let $v \in V$, Let $w = \sum_{i=1}^{m} \langle v, w_i\rangle w_i$, $w_i \in W$ and $x = v - w$

Then $\langle x, w_j\rangle = 0$ for all $j = 1(1)m$.

$\therefore \langle x, w\rangle = \langle x, \beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_m w_m\rangle$

$= \bar{\beta}_1 \langle x, w_1\rangle + \bar{\beta}_2 \langle x, w_2\rangle + \cdots + \bar{\beta}_m \langle x, w_m\rangle$

$= 0$ for all $w \in W$

$\therefore x \in W^\perp$. So $v = w + x \in W + W^\perp$

$\Rightarrow V \subseteq W + W^\perp \Rightarrow V = W + W^\perp$

Let $y \in W \cap W^\perp \Rightarrow \langle y, w\rangle = 0$ for all $w \in W$, $y \in W^\perp$

$\Rightarrow \langle y, y\rangle = 0$ as $y \in W$

$\Rightarrow y = 0$ $\therefore W \cap W^\perp = \{\theta\}$

Hence $V = W \oplus W^\perp$.

Result: If W is a subspace of a finite dimensional inner product space V, then $(W^{\perp})^{\perp} = W$.

$V = W \oplus W^{\perp}$

Let $x \in W$, $x \in W^{\perp}$. Then $x \in W^{\perp} \Rightarrow \langle x, y \rangle = 0$ $\forall y \in W$.

$\Rightarrow \langle x, w \rangle = 0 \quad \forall x \in W^{\perp}$

$\Rightarrow w \in (W^{\perp})^{\perp} \Rightarrow W^{+} \subseteq (W^{\perp})^{\perp}$

Let $v \in (W^{\perp})^{\perp}$ then $v = w + w'$, $w \in W$, $w' \in W^{\perp}$

$\Rightarrow 0 = \langle w', v \rangle = \langle w', w + w' \rangle = \langle w', w \rangle + \langle w', w' \rangle$

$= \langle w', w' \rangle$

So $w' = 0 \Rightarrow v = w \in W$.

i.e, $(W^{\perp})^{\perp} \subseteq W \Rightarrow W = (W^{\perp})^{\perp}$.

7. Let T be a linear operator on a finite dimensional inner product space V and suppose T has an igen vector. Show that T^* also has an eigenvector.

Ans: Let $T(v) = \alpha v$, $v \neq \theta$. Then for any $x \in V$

$0 = \langle \theta, x \rangle = \langle (T - \alpha I) v, x \rangle = \langle v, (T - \alpha I)^* x \rangle$

$= \langle v, (T^* - \bar{\alpha} I) x \rangle$

If $T^* - \bar{\alpha} I$ is onto, then for any $v' \in V$,

$v' = (T^* - \bar{\alpha} I)(x')$ for some $x' \in V$.

Now $0 = \langle v, v' \rangle \ \forall v' \in V \Rightarrow 0 = \langle v, v \rangle \Rightarrow v = \theta$, a contradiction

So, $T^* - \bar{\alpha} I$ is not onto. Therefore, $T^* - \bar{\alpha} I$ is not 1-1 as V is finite dimensional. Therefore, $\operatorname{Ker}(T^* - \bar{\alpha} I) \neq \{\theta\} \Rightarrow \exists\, 0 \neq y \in \operatorname{Ker}(T^* - \bar{\alpha} I) \Rightarrow (T^* - \bar{\alpha} I) y = 0$

$\Rightarrow T^*(y) = \bar{\alpha} y,\ y \neq 0 \Rightarrow y$ is an eigen vector of T^* with eigen value $\bar{\alpha}$.

Also $T^*\left(\frac{y}{\|y\|}\right) = \bar{\alpha}\left(\frac{y}{\|y\|}\right) \Rightarrow T^*(z) = \bar{\alpha} z,\ z = \frac{y}{\|y\|}$ is a unit vector.

8. Let V be a finite dimensional inner product space. Let T be a linear operator on V. Prove that $\operatorname{Ker} T = \operatorname{Ker} = T^*T$.

Ans: Let $x \in \operatorname{Ker} T$. Then $Tx = 0 \Rightarrow T^*Tx = 0$

$\Rightarrow x \in \operatorname{Ker} TT^* \Rightarrow \operatorname{Ker} T \subset \operatorname{Ker} T^*T$ —— ①

Let $x \in \operatorname{Ker} T^*T$. Then $\langle x, T^*Tx\rangle = 0$

$\Rightarrow \langle Tx, Tx\rangle = 0 \Rightarrow Tx = 0 \Rightarrow x \in \operatorname{Ker} T$

$\Rightarrow \operatorname{Ker} T^*T \subset \operatorname{Ker} T$ —— ②

From ① and ② $\operatorname{Ker} T = \operatorname{Ker} T^*T$.

9. Let W be a subspace of finite dimensional inner product space V such that $V = W \oplus W^\perp$. Show that the adjoint of P_W is itself.

Ans: Let $\{W_1, W_2, W_3, \dots, W_m\}$ be an orthonormal basis of W. Now $\langle v - \sum_{i=1}^{m} \langle v, w_i\rangle w_i, w_j\rangle = 0\ \forall j$

$v - \sum_{i=1}^{m} \langle v, w_i\rangle w_i \in W^\perp \Rightarrow P_W\left(v - \sum_{i=1}^{m} \langle v, w_i\rangle w_i\right) = 0$

$\Rightarrow P_W(v) = \sum_{i=1}^{m} \langle v, w_i\rangle w_i \Rightarrow \langle P_W(v), v\rangle = \langle \sum_{i=1}^{m} \langle v, w_i\rangle w_i, v\rangle$

$= \sum_{i=1}^{m} |\langle v, w_i\rangle|^2$

Also $\langle v, P_W(v)\rangle = \langle v, \sum_{i=1}^{m} \langle v, w_i\rangle w_i\rangle = \sum_{i=1}^{m} |\langle v, w_i\rangle|^2\ \forall v \in V$

$\Rightarrow P_W^* = P_W$. [Such an operator is called self adjoint].

10. Give an example of a linear operator T such that $\ker T \neq \ker T^*$.

Ans: Let $V = M_2(\mathbb{C})$. Let $A = E_{12} \in V$. Let T be a linear operator on V defined by $T(B) = AB$. Then $T^*(B) = A^*B$. Here $B \in \ker T \Leftrightarrow AB = 0 \Leftrightarrow E_{12}B = 0 \Leftrightarrow$ 2nd row of B is zero. Also $B \in \ker T^* \Leftrightarrow A^*B = 0 \Leftrightarrow E_{12}^*B = 0$ $\Leftrightarrow$ 1st row of B is zero. So $\ker T \neq \ker T^*$.

Defn. (Normal operator): Let T be a linear operator on an inner product space V. We say T is normal iff $TT^* = T^*T$.

If A is $n\times n$ matrix over $\mathbb{C}$, Then A is said to be normal if $AA^* = A^*A$. Suppose T is normal Then $TT^* = T^*T \Leftrightarrow [TT^*]_\beta = [T^*T]_\beta$ $\forall$ orthonormal basis β of V.

$\Leftrightarrow [T]_\beta [T^*]_\beta = [T^*]_\beta [T]_\beta$

$\Leftrightarrow [T]_\beta [T]^*_\beta = [T]^*_\beta [T]_\beta$

$\Leftrightarrow [T]_\beta$ is normal $\forall$ orthonormal basis β of V.

11.

11. Let T be a linear operator on a finite dimensional complex inner product space V. Then T is normal iff $\exists$ an orthonormal basis β of V consisting of eigen vectors of T

Ans: Let T be normal. By fundamental theorem of algebra, the characteristic polynomial of T splits. $\exists$ an orthonormal basis β of V such that $[T]_\beta = A$ is upper triangular. Let $\beta = \{v_1, v_2, \ldots, v_n\}$. Since A is upper triangular, $T(v_1) = \lambda v_1 \Rightarrow v_1$ is an eigen vector of T. Suppose $v_1, v_2, \ldots, v_{k-1}$ are eigen vectors of T. Let $T(v_1) = \lambda_1 v_1, \ldots, T(v_{k-1}) = \lambda_{k-1} v_{k-1}$

$\Rightarrow T^*(v_1) = \bar{\lambda}_1 v_1, \ldots, T^*(v_{k-1}) = \bar{\lambda}_{k-1} v_{k-1}$.

Since A is upper triangular, $T(v_k) = a_{1k} v_1 + \cdots + a_{kk} v_k$. Also $a_{jk} = \langle T(v_k), v_j \rangle = \langle v_k, T^* v_j \rangle$

$= \langle v_k, \bar{\lambda}_j v_j \rangle = \lambda_j \langle v_k, v_j \rangle$.

So, $T(v_k) = a_{kk} v_k$ as $\langle v_k, v_j \rangle = 0 \ \forall j \neq k$.

$\Rightarrow v_k$ is an eigen vector of T.

By induction on k, β is an orthonormal basis of V of eigen vectors of T.

Conversely, let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis of V such that $T(v_i) = \lambda_i v_i \ \forall i$

Then $[T]_\beta = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ 0 & \lambda_2 \cdots & 0 \\ 0 & 0 \cdots & \lambda_n \end{bmatrix} = A$.

$\Rightarrow [T^*]_\beta = [T]^*_\beta = A^* =$ diagonal matrix

$\Rightarrow T^*(v_i) = \bar{\lambda}_i v_i \ \forall i \Rightarrow [TT^*]_\beta = [T]_\beta [T^*]_\beta = AA^*$

$= A^* A = [T^*]_\beta [T]_\beta = [T^* T]_\beta \Rightarrow TT^* = T^* T$

$\Rightarrow T$ is normal.

Result 1. The above result need not be true if T is linear operator on a real inner product space.

Let $T: R^2 \to R^2$ defined by $T(a_1, a_2) = (a_1 \cos\theta - a_2 \sin\theta, a_1 \sin\theta + a_2 \cos\theta)$, $0 < \theta < \pi$.

Then T is a linear operator on R^2, called rotation by θ. Here $T(1,0) = (\cos\theta, \sin\theta)$, $T(0,1) = (-\sin\theta, \cos\theta)$

Let $\beta = \{e_1 = (1,0), e_2 = (0,1)\}$.

Then $[T]_\beta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A$, $A^* = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

and $AA^* = I = A^* A \Rightarrow T$ is normal, β is an orthonormal basis of $V = R^2$. Characteristic polynomial of A is $x^2 - 2x\cos\theta + 1$ which has no ~~real~~ roots in $R \Rightarrow V = R^2$ which is an inner product space over R and has no orthonormal basis of eigen vector of T.

Defn. (Self-adjoint operator): A linear operator T on an inner product space V is called Self adjoint if $T = T^*$. If A is an $n \times n$ matrix over $\mathbb{C}$, then A is called Hermitian (or Self-adjoint) if $A = A^*$.

Not that T is Self adjoint. $\Leftrightarrow T = T^* \Leftrightarrow [T]_\beta = [T^*]_\beta$, $\forall$ orthonormal basis β of V.

$\Leftrightarrow [T]_\beta = [T]_\beta^*$ $\forall$ orthonormal basis β of V.

$\Leftrightarrow [T]_\beta$ is self-adjoint or Hermitian.

12. If A is real symmetric matrix then $A = A^t \Rightarrow A^* = (A^t)^*$ $= A$
$\Rightarrow A$ is self adjoint $\Rightarrow AA^* = A^2 = AA = A^*A$
$\Rightarrow A$ is normal.
However, if A is complex symmetric matrix, then A need not be normal.
Let $A = \begin{bmatrix} i & i \\ i & 1 \end{bmatrix} = A^t$

$A^* = \begin{bmatrix} -i & -i \\ -i & 1 \end{bmatrix}$. Then $AA^* \neq A^*A \Rightarrow A$ is not normal.

13. Let T be a self adjoint operator on a finite dimensional inner product space V. Then every eigen value of T is real.

Ans: Let $Tx = \lambda x$, $x \neq \theta \Rightarrow T^*x = \bar{\lambda} x$

$\Rightarrow Tx = \bar{\lambda} x$ as $T = T^*$

$\Rightarrow \lambda x = \bar{\lambda} x \Rightarrow \lambda = \bar{\lambda}$ as $x \neq \theta$

$\Rightarrow \lambda$ is real. $\Rightarrow$ every eigen value of T is real.

14. Suppose T is a linear operator on a finite dimensional real inner product space V. If T is self adjoint, show that the characteristic polynomial of T splits in $\mathbb{R}$.

Ans: Let $\dim V = n$. Let β be an orthonormal basis of V. Let $A = [T]_\beta$. Then $A^* = [T]^*_\beta$

$= [T^*]_\beta = \overline{[T]_\beta} = A.$

So, A is self adjoint $\Rightarrow T$ is self adjoint.

Define, $T_A : \mathbb{C}^n \to \mathbb{C}^n$ s.t. $T_A(x) = Ax \; \forall x \in \mathbb{C}^n$.

Then T_A is a linear operator on $\mathbb{C}^n$.

Let γ be the standard orthonormal basis of $\mathbb{C}^n$.

Then $[T_A]_\gamma = A$

as $T_A \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} = a_{11}e_1 + \cdots + a_{n1}e_n$

$\vdots$

$T_A \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} = a_{1n}e_1 + \cdots + a_{nn}e_n.$

So, $[T_A]_\gamma^* = A^* = A$

$\Rightarrow [T_A^*]_\gamma = A = [T_A]_\gamma$

$\Rightarrow T_A^* = T_A \Rightarrow T_A$ is self adjoint.

Therefore, eigen values of T_A are real.

But characteristic polynomial of T_A splits in $\mathbb{C}$.

Since eigen values of T_A are real, characteristic polynomial of T_A splits in $\mathbb{R}$.

Also, characteristic polynomial of T_A

= characteristic polynomial of A

= characteristic polynomial of T

Thus, the characteristic polynomial of T splits in R

14. Let T be a linear operator on a finite dimensional real inner product space V. Then T is self-adjoint iff $\exists$ an orthonormal basis β of V consisting of eigen vectors of T.

Ans: Let T be self adjoint. Then the characteristic polynomial of T splits in R and $\exists$ an orthonormal basis β of V such that $[T]_\beta$ is an upper triangular matrix A. Let $\beta = \{v_1, v_2, \ldots, v_n\}$.

But $A^* = [T]_\beta^* = [T^*]_\beta = [T]_\beta = A$

$$\Rightarrow A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \bar{a}_{11} & 0 & \cdots & 0 \\ \bar{a}_{12} & \bar{a}_{21} & \cdots & 0 \\ - & - & - & - \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & \bar{a}_{nn} \end{bmatrix}$$

$\Rightarrow$ A is diagonal matrix

$\Rightarrow$ Each v_i is an eigen vector of T

Conversely, let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis of V s.t. $T(v_i) = \lambda_i v_i$, $\lambda_i \in R$.

$$\Rightarrow [T]_\beta = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = A$$

$$\Rightarrow [T^*]_\beta = \operatorname{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n)$$

$$= \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = A = [T]_\beta$$

$\Rightarrow T = T^* \Rightarrow$ T is self adjoint.

Ex-15. Let T be a linear operator on a finite dimension -al inner product space V. Show that if T is self adjoint, so is T_W.

Ans: Let $x, y \in W$. Now $\langle x, T_W(y)\rangle = \langle x, T(y)\rangle$

$$= \langle T^* x, y\rangle = \langle Tx, y\rangle = \langle T_W x, y\rangle$$

$$= \langle x, T_W^* y\rangle \ \forall\ x, y \in W$$

$$\Rightarrow T_W(y) = T_W^*(y) \ \forall\ y \in W$$

$$\Rightarrow T_W = T_W^* \Rightarrow T_W \text{ is self adjoint.}$$

P-16) Let T be a self adjoint operator on a finite dimensional inner product space V. Prove that for all $x \in V$, $\|Tx \pm ix\|^2 = \|Tx\|^2 + \|x\|^2$.
Deduce that $T - iI$ is inveertiable and $[(T-iI)^{-1}]^*$ $= (T+iI)^{-1}$.

Ans: $\|Tx + ix\|^2 = \langle Tx + ix, Tx + ix\rangle$

$$= \|Tx\|^2 + \|x\|^2 - i\langle Tx, x\rangle + i\langle x, Tx\rangle$$

$$= \|Tx\|^2 + \|x\|^2 - i\langle x, T^* x\rangle + i\langle x, T^* x\rangle \text{ as } T = T^*$$

$$= \|Tx\|^2 + \|x\|^2.$$

Similarly, the other epuality can be proved

Let $x \in \operatorname{Ker}(T - iI)$

Then $(T - iI)x = 0 \Rightarrow Tx - ix = 0$

$$\Rightarrow \|Tx\|^2 + \|x\|^2 = 0$$

$$\Rightarrow \|x\| = 0 \Rightarrow x = 0 \Rightarrow \operatorname{Ker}(T - iI) = \{0\}$$

$\Rightarrow T - iI$ is 1-1 and so onto

$\Rightarrow T - iI$ is invertible.

Now $[(T - iI)^{-1}]^* = [(T - iI)^*]^{-1} = [T^* + iI]^{-1} = (T + iI)^{-1}$

Ex-17. For $z \in \mathbb{C}$, define $T_z : \mathbb{C} \to \mathbb{C}$ by $T(z) = zu$.
Find z such that T_z is self-adjoint.

Ans: Suppose $T_z = T_z^*$

Then $T_z(v) = T_z^*(v)$

$\Rightarrow \quad zv = \bar{z}v \;\forall\, v \in V$

$\Rightarrow z = \bar{z} \Rightarrow z$ is real.

Conversely, Let z be real number

Then $T_z(v) = zv = \bar{z}v = T_z^*(v) \;\forall\, v \in \mathbb{C}$

$\Rightarrow T_z = T_z^* \Rightarrow T_z$ is self adjoint.

Defn. (Unitary and Orthogonal Operators):
Let T be a linear operator on an inner product space V. Then T is called unitary if $TT^* = T^*T = I$ and $F = \mathbb{C}$. If $F = \mathbb{R}$ and ~~of~~ $TT^* = T^*T = I$, then T is called orthogonal.

Similarly, A square matrix A is called unitary if $AA^* = A^*A = I$ and orthogonal if $AA^t = A^tA = I$

Ex-18. Let T be a linear operator on a finite-dimensional inner product space V. Then T is unitary iff $\|Tx\| = \|x\| \;\forall\, x \in V$.

Ans: Let T be unitary. Then $TT^* = T^*T = I$

Now $\langle Tx, Tx\rangle = \langle x, T^*Tx\rangle = \langle x, Ix\rangle = \langle x, x\rangle$

$\Rightarrow \|Tx\|^2 = \|x\|^2 \Rightarrow \|Tx\| = \|x\| \ \forall x \in V.$

Conversely,

Now $\langle x, x\rangle = \|x\|^2 = \|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle$

$\Rightarrow \langle x, (I - T^*Tx)\rangle = 0 \ \forall x \in V$

$\Rightarrow (I - T^*T)x = 0 \ \forall x \in V$

$\Rightarrow T^*Tx = x \ \forall x \in V$

$\Rightarrow T^*T = I$. Since V is finite dimensional, $TT^* = I$. So T is unitary.

Result: Let T be a unitary operator on a finite dimensional inner product space. The eigen values of T have absolute value 1.

Ans: Let λ be an eigen value of T.

Then $Tx = \lambda x$, $x \neq \theta$, $x \in V$.

$\Rightarrow \|Tx\| = \|\lambda x\| = |\lambda| \|x\|$

$\Rightarrow \|x\| = |\lambda| \|x\| \Rightarrow |\lambda| = 1$ as $\|x\| \neq \theta$.

Ex-19) Let T be a linear operator on a finite dimensional vector space $V(F)$. Let $p(x)$ be the minimal polynomial of T. Then T is diagonalisable iff $p(x) = (x - c_1)(x - c_2) \cdots (x - c_k)$, where $c_1, c_2, \ldots, c_k \in F$ are distinct.

Ans: Let $p(x) = (x-c_1)(x-c_2) \cdots (x-c_k)$

let $f_i(x) = \dfrac{p(x)}{(x-c_i)}$, $i = 1(1)k$.

Then g.c.d. $(f_1, f_2, \ldots, f_k) = 1$

$\Rightarrow \exists\, g_1, g_2, \ldots, g_k \in F[x]$ s.t. $g_1 f_1 + g_2 f_2 + \cdots + g_k f_k = 1$

$\therefore g_1(T) f_1(T) + \cdots + g_k f_k(T) = I$.

Let $v \in V$. Then $v = g_1(T) f_1(T) v + \cdots + g_k(T) f_k(T) v$

$\Rightarrow f_1(T) v = (T - c_2 I) \cdots (T - c_k I)(v)$

$\Rightarrow (T - c_1 I) f_1(T)(v) = (T - c_1 I)(T - c_2 I) \cdots (T - c_k I)(v)$

$= p(T)(v) = \theta$.

$\Rightarrow f_1(T)(v) \in \mathrm{Ker}\,(T - c_1 I) = W_1$

$\Rightarrow g_1(T) f_1(T)(v) \in W_1$ [as $w_1 \in W_1 \Rightarrow T(w_1) = c_1 w_1$

$\Rightarrow T^r(w_1) = c_1^r w_1 \in W_1$]

Similarly, $g_i(T) f_i(T)(v) \in W_i \;\; \forall i$

$\Rightarrow v \in W_1 + W_2 + \cdots + W_k$

$\Rightarrow V = W_1 + W_2 + \cdots + W_k$

$\Rightarrow \dim V = \dim W_1 + \cdots + \dim W_k \Rightarrow T$ is diagonalisable.

Conversely, Let T be diagonalisable then $\exists$ a basis $\beta = \{v_1, v_2, \ldots, v_n\}$ of V such that each v_i is an eigen vector of T.

Now $(T - c_1 I) \cdots (T - c_k I)(v_i) = 0 \;\; \forall i$

as each v_i belongs to some eigen space.

$W_j = \mathrm{Ker}\,(T - c_j I)$. $\therefore p(x) = (x-c_1) \cdots (x-c_k)$ is the minimal polynomial.